Brush-up Math Course

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Lecture Outline

1. Introduction - basic math concepts
2. **Linear algebra**
3. Calculus
4. Optimisation

Lecture 2: Linear algebra

September 2021

Lecture Outline

Matrices and vectors

Elementary matrix operations

Matrix characteristics

Matrix operations

Solving system of linear equations

Eigenvalues & eigenvectors

Introduction

Matrix

- **Matrix** is a rectangular array of numbers, symbols, or expressions, arranged in m-rows and n-columns.

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

- a_{ij} is an *entry* of a matrix A which lies in the i-th row and j-column.

- An example of numeric matrix:

$$B_{2\times 2} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}_{2\times 2}$$

Introduction

Vector

- **Column Vector** is an $m \times 1$ matrix, that is matrix consisting of single column of m elements.

- **Row Vector** is an $1 \times n$ matrix, that is matrix consisting of single row of n elements.

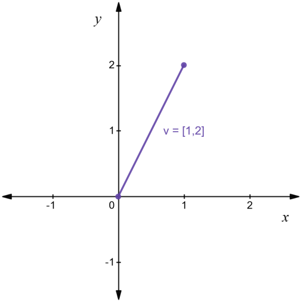
$$\boldsymbol{x_{m\times 1}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}_{m\times 1} \qquad \boldsymbol{x_{1\times n}} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}_{1\times n}$$

- $x_1, x_2, ..., x_m$ are called coordinates of a vector, which can be used to represent vector in $\mathbb{R}^m$ dimensional space

Introduction

Vector

- Vectors, unlike general matrices, are easily interpreted geometrically

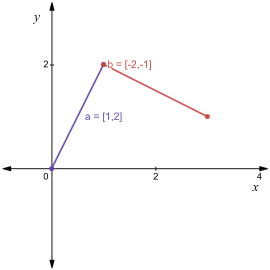


Each coordinate of a vector correspond to the movement in the plane e.g. v = [1,2] = 1 step on x- and 2 steps on y-axis

- *Norm* (length) of a vector is given by:

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

- Two vectors are called *orthogonal* if the angle between them is 90^0, e.g.:



- Vectors are orthogonal if $a \cdot b = 0$ (see matrix multiplication in the next slides)

Typical matrices

Zero matrix

- *Zero matrix* is a matrix which each entry is zero and denoted by $\mathbf{0}$, e.g.:

$$\mathbf{0}_{m\times n} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{m\times n}$$

Typical matrices

Squared matrix

- *Squared matrix* is a matrix which number of rows equals number of columns ($n = m$). They are usually denoted by A_m, e.g.:

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}_{m\times m} \qquad B_3 = \begin{bmatrix} 1 & 4 & 3 \\ -2 & 5 & 7 \\ -1 & -3 & -4 \end{bmatrix}_{3\times 3}$$

- The entries $a_{11}, a_{22}, \dots, a_{mm}$ of a squared matrix A are called *diagonal entries* and form the diagonal of A

Typical matrices

Diagonal matrix

- *Diagonal matrix* is a squared matrix which all entries lying out of the diagonal are equal zero, e.g.:

$$A_m = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{mm} \end{bmatrix}_{m\times m} \qquad B_3 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$$

- Diagonal matrix could be also denoted by:

$$A = diag(a_1, a_2, \dots, a_m)$$

- If all diagonal entries are equal $(a_{11} = a_{22} = \dots = a_{mm} = a)$, then the matrix is called a *scalar matrix*.

Typical matrices

Identity matrix

- *Identity matrix* is a diagonal matrix which all diagonal entries are equal 1: $a_{11} = a_{22} = \ldots = a_{mm} = 1$. Identity matrices are usually denoted by $\boldsymbol{I}_m$, e.g.:

$$\boldsymbol{I}_m = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}_{m\times m} \qquad \boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Typical matrices

Triangular matrix

- A squared matrix, which all entries lying below/above the diagonal are 0, is called *upper/downer triangular matrix*:

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{mm} \end{bmatrix}_{m\times m} \qquad B_3 = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -4 \end{bmatrix}_{3\times 3}$$

Upper triangular matrix

$$A_m = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}_{m\times m} \qquad B_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 5 & 0 \\ -1 & -3 & -4 \end{bmatrix}_{3\times 3}$$

Downer triangular matrix

Elementary matrix operations

Introduction

- *Elementary matrix operations* help to find an inverse of a matrix or solve system of linear equations (see more later)
- They can be performed on rows or columns
- There are 3 types of elementary matrix operations:
  - Row/column switching
  - Row/column multiplication
  - Row/column addition
- The next slides focus on elementary row operation

Elementary matrix operations

Row switching

- *Row switching* - a row within the matrix is switched with another row, e.g.:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{11} & b_{12} & b_{13} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -2 & 4 \end{bmatrix} R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 3 \\ -2 & 4 \\ 2 & 5 \end{bmatrix}$$

Elementary matrix operations

Row multiplication

- *Row multiplication* - each element of a row is multiplied by a non-zero constant, e.g.:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} k \cdot R_1 \rightarrow R_1 \begin{bmatrix} b_{11} \cdot k & b_{12} \cdot k & b_{13} \cdot k \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -2 & 4 \end{bmatrix} 2 \cdot R_2 \rightarrow R_2 \begin{bmatrix} 1 & 3 \\ 4 & 10 \\ -2 & 4 \end{bmatrix}$$

Elementary matrix operations

Row addition

- *Row addition* - add to a row a multiplied values of another row, e.g.:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} R_1 + k \cdot R_2 \rightarrow R_1 \begin{bmatrix} b_{11} + b_{21} \cdot k & b_{12} + b_{22} \cdot k & b_{13} + b_{21} \cdot k \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -2 & 4 \end{bmatrix} R_3 + (-2) \cdot R_2 \rightarrow R_3 \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -2 + (-2) \cdot 2 & 4 + (-2) \cdot 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 10 \\ -6 & -6 \end{bmatrix}$$

Matrix

Linear dependence

- *Column/row is linearly dependent* if it is a linear combination of other columns/rows
- Linear dependence can be checked by conducting elementary operations

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} R_3 + R_2 \to R_3 \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$R_3 - R_1 \to R_3 \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- R_3 became a zero row vector $\implies$ R_3 is linearly dependent

Matrix

Rank of a matrix

- *Rank of a matrix* - number of linearly independent columns/rows
- Rank of a matrix is denoted by $rank(A)$
- Rank of a matrix cannot be bigger than number of columns and rows: $A_{m\times n} \longrightarrow rank(A) \leq min(m,n)$
- If $rank(A) = min(m,n)$, then we say that A is a *full rank*
- So, matrix A from the previous slide has rank 2 and is not full rank

- *Determinant* of a matrix A, denoted by $det(A)$, is a scalar value representing the area of expansion of the transformation induced by a matrix

- If $det(A) \neq 0 \longrightarrow A$ is *full rank*

- Calculation of determinant for bigger matrices is not trivial

- The next slides present method of calculating determinant for 2×2 and 3×3 matrices

Matrix

Determinant of 2×2 matrix

- For a matrix 2×2 determinant is calculated in the following way:

$$det(A_2) = det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$$

- e.g.:

$$det(\begin{bmatrix} 1 & -5 \\ 2 & -1 \end{bmatrix}) = 1 \cdot (-1) - (-5) \cdot 2 = -1 + 10 = 9$$

- Calculation for a 3×3 matrix is more complicated:

$$det(A_3) = det(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix})$$

$$= a \cdot det(\begin{bmatrix} e & f \\ h & i \end{bmatrix}) - b \cdot det(\begin{bmatrix} d & f \\ g & i \end{bmatrix}) + c \cdot det(\begin{bmatrix} d & e \\ g & h \end{bmatrix})$$

$$det(B) = det(\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 1 \\ 1 & 2 & 0 \end{bmatrix}) =$$

$$= 1 \cdot det(\begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}) - 2 \cdot det(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) + (-3) \cdot det(\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}) = -6$$

- *Trace of a matrix*, denoted by $tr(A)$, is a sum of the diagonal entries.

$$tr(A_m) = tr(\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}) = a_{11} + \dots + a_{mm} = \sum_{i=1}^{m} a_{ii}$$

- e.g.:

$$tr(\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 3 \\ -1 & 3 & -3 \end{bmatrix}) = 1 + (-2) + (-3) = -4$$

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Operations with matrix and scalar
Addition

- *Addition:*

$$A_{m\times n} + x = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} + x =$$

$$= \begin{bmatrix} a_{11}+x & a_{12}+x & \dots & a_{1n}+x \\ a_{21}+x & a_{22}+x & \dots & a_{2n}+x \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+x & a_{m2}+x & \dots & a_{mn}+x \end{bmatrix}_{m\times n}$$

Operations with matrix and scalar

Multiplication

- *Multiplication:*

$$A_{m\times n} \cdot x = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} \cdot x =$$

$$= \begin{bmatrix} a_{11}\cdot x & a_{12}\cdot x & \dots & a_{1n}\cdot x \\ a_{21}\cdot x & a_{22}\cdot x & \dots & a_{2n}\cdot x \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\cdot x & a_{m2}\cdot x & \dots & a_{mn}\cdot x \end{bmatrix}_{m\times n}$$

Matrix operation

Transposition

- *Transposition* changes columns into rows and rows into columns ($a_{ij} \rightarrow a_{ji}$)

$$A^T_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}^T_{m\times n} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}_{n\times m} = B_{n\times m}$$

$$C^T_{3\times 2} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -2 & 4 \end{bmatrix}^T_{3\times 2} = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 5 & 4 \end{bmatrix}_{2\times 3} = D_{2\times 3}$$

- Transposition does not change determinant ($det(A) = det(A^T)$), trace ($tr(A) = tr(A^T)$) and rank ($rank(A) = rank(A^T)$)

Matrix operation

Transposition

- Transposition can be denoted by both: A^T and A'
- Matrix is called *symmetric* if $A = A^T$, e.g.:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & 5 \\ 2 & 5 & -2 \end{bmatrix}_{3\times 3} = A^T$$

- Only squared matrix can be symmetric
- Rules of transposition:
  - $(A^T)^T = A$
  - $(A+B)^T = A^T + B^T$
  - $(AB)^T = B^T A^T$

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Matrix operation

Addition

- One can add $A_{m\times n}$ to $B_{x\times y}$ if they have the same shape
- If we can add matrices, then we call them *conformable*
- It is an element by element $(a_{ij} + b_{ij})$ operation.

$$A_{m\times n} + B_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}_{m\times n}$$

Matrix operation

Addition

- Let's take A and B:

$$A_{3\times3}=\begin{bmatrix}1 & 0 & 2\\ -1 & 1 & 3\\ 2 & -1 & -2\end{bmatrix},\quad B_{3\times3}\begin{bmatrix}1 & -2 & 1\\ 5 & -3 & -1\\ 2 & 3 & -2\end{bmatrix}$$

- Then, the sum $A+B$ is:

$$A_{3\times3}+B_{3\times3}=\begin{bmatrix}2 & -2 & 3\\ 4 & -2 & 2\\ 4 & 2 & -4\end{bmatrix}_{3\times3}$$

Matrix operation

Addition - rules

- Rules for matrix addition and multiplication by a scalar:
  - $(A+B)+C=(A+B+C)$
  - $A+B=B+A$ - addition is commutative
  - $A+\mathbf{0}= \; A$ - zero matrix is a neutral element
  - $A-A=\mathbf{0}$
  - $(\alpha+\beta)A=\alpha A+\beta A$
  - $\alpha(A+B)=\alpha A+\alpha B$

Matrix operation

Multiplication

- Matrix multiplication is different from "standard" multiplication
- Two matrices can be multiplied if number of columns of the first one is equal to number of rows of the second one
- $A_{m\times n}B_{n\times y} \longrightarrow C_{m\times y}$ - new matrix has a number of rows the same like the first one and number of columns the same like the second one
- c_{ij} = dot product of row i of matrix A and column j of matrix B:

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \dots \\ b_{yj} \end{bmatrix} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \dots + a_{in} \cdot b_{yj}$$

Matrix operation

Multiplication

- Let's take A and B:

$$A_{3\times 2} = \begin{bmatrix} 6 & -8 \\ 1 & -2 \\ -4 & 5 \end{bmatrix}, \quad B_{2\times 3} = \begin{bmatrix} 1 & -3 & 2 \\ 5 & -1 & 1 \end{bmatrix}$$

- then:

$$c_{11} = \begin{bmatrix} 6 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 6 \cdot 1 + (-8) \cdot 5 = -34...$$

- As a result:

$$A_{3\times 2}B_{2\times 3} = C_{3\times 3} = \begin{bmatrix} -34 & -26 & 4 \\ -9 & -1 & 0 \\ 21 & 7 & -3 \end{bmatrix}$$

Matrix operation

Multiplication

- Matrix multiplication is NOT commutative - even though AB and BA exist, AB is not necessary equal to BA
- Let's take A, B:

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ -5 & 2 \end{bmatrix}$$

- In this case, both AB and BA exist
- Let's check if $AB = BA$

Matrix operation

Multiplication

- Then:

$$AB = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 17 & -7 \\ 3 & 4 \end{bmatrix}$$

- But:

$$BA = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ -2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 & -7 \\ 3 & 0 & -9 \\ -9 & 8 & 17 \end{bmatrix}$$

Matrix operation

Multiplication - rules

- Rules for matrix multiplication:
  - $(AB)C = A(BC)$ - associative
  - $AA = A^2$ - multiplication of square matrix with itself can be written in power form
  - $(\alpha A)B = A(\alpha B) = \alpha(AB)$ - associative scalar multipl.
  - $A(B + C) = AB + AC$ - distributive law
  - $AB \neq BA$ - not commutative
  - $A\boldsymbol{I} = \boldsymbol{I}A = A$ - identity matrix is a neutral element
  - $det(AB) = det(A) \cdot det(B)$

Matrix operation

Hadamard product

- The *Hadamard product* (denoted by $\odot$) - multiply two matrices entry by entry.

$$A_{m\times n} \odot B_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \odot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11}\cdot b_{11} & a_{12}\cdot b_{12} & \dots & a_{1n}\cdot b_{1n} \\ a_{21}\cdot b_{21} & a_{22}\cdot b_{22} & \dots & a_{2n}\cdot b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\cdot b_{m1} & a_{m2}\cdot b_{m2} & \dots & a_{mn}\cdot b_{mn} \end{bmatrix}_{m\times n}$$

Matrix operation

Inverse of a matrix

- Let's take $a \in \mathbb{R} \setminus \{0\}$. We called b an *inverse* of a if $ab = 1$
- Usually we denote inverse of a number as a^{-1}
- In matrix world an equivalent of 1 is identity matrix I
- So, inverse of a square matrix A is a square matrix B such that $BA = AB = I$
- B is usually denoted by A^{-1}
- If matrix has an inverse, then it is called *invertible* or *non-singular*
- Only square matrices can be invertible

Matrix operation

Inverse of a matrix - example

- Let's take the following matrix A:

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$$

- Then, take matrix A^{-1}:

$$A^{-1} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$$

- Check if $AA^{-1} = I$

$$AA^{-1} = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix operation

Inverse of a matrix

- Not every non-zero squared matrix has a inverse, e.g.:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

- Then B is an inverse if $AB = \boldsymbol{I}$:

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- So we have to solve the following equation:

$$0 \cdot b_{12} + 0 \cdot b_{22} = 0 \neq 1 \quad \forall b_{12}, b_{22} \in \mathbb{R}^2$$

Matrix operation

Inverse of a matrix

- In the above example an inverse of matrix A does not exist
- Notice: A is not $full\ rank \longrightarrow det(A) = 0$
- Matrix is invertible $\rightarrow det(A) \neq 0$ (or A is *full rank*)
- If A is invertible then $det(A^{-1}) = \frac{1}{det(A)}$
- Let A and B be invertible, then:
  - $(A^{-1})^{-1} = A \longrightarrow A^{-1}$ is invertible and its inverse is A
  - AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
  - A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$
  - if c is a constant and $c \neq 0$, then $(cA)^{-1} = c^{-1}A^{-1}$

Matrix operation

Inverse of a matrix

- Finding a inverse of the matrix is usually not a straightforward task
- There exists an easy formula for 2×2 matrix inversion:

$$A_2^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{bmatrix}$$

Matrix operation

Inverse of a matrix

- In case of bigger matrices we need to use elementary operations
- To find inverse of matrix $A_{n \times n}$, assuming that it exists (so $det(A) \neq 0$):
  1. Form matrix $(A, I)_{n \times 2n}$
  2. Using an elementary operations try to transform this matrix into $(I : B)_{n \times 2n}$
  3. Then $B = A^{-1}$
- Example: $(A, I) = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$

- Matrices are useful to solve system of n linear equations
- Let's consider a system of m equations and n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

- The equations are linear in the unknowns
- Some $a_{i,j}$ can be equal 0

Solving system of linear equations

Solutions

- System of equations is:
  - *consistent* - has at least one solution:
    - *unique* - has exactly one solution
    - has infinitely many solutions
  - *inconsistent* - has no solution at all
- If $m < n$, then system is inconsistent or has infinite many solutions
- If $m \geq n$, then system can have any of 3 possible solutions

Solving system of linear equations

Matrix notation

- System of equations can be rewritten using matrix notation:

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b}$$

- Or shorter:

$$Ax = b$$

Solving system of linear equations

Gaussian elimination

- Gaussian elimination is a method to solve a system of linear equations
- The method consists of following steps:
  - Form matrix $(A, b)_{m,n+1}$
  - Using elementary operations to modify the matrix until the lower left-hand corner of the matrix is filled with zeros, as much as possible
  - Then the last right column vector is a solution to the system of equations

Solving system of linear equations

Gaussian elimination

- Consider following system of equations:

$$\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$$

- Then the augmented matrix $(A, b)_{3,4}$ is:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$$

- Use elementary operations to find the solution

Solving system of linear equations

Matrix inversion

- Assume that $m = n$ - number of equations is equal to the number of variables
- Then, if A is *full-rank* (so is invertible):

$$x = A^{-1}b$$

- It allows to find the solution to the system of linear equations using the inverse of parameters matrix
- So, if A is a squared matrix and is invertible then the system of equation has a unique solution

- If there is a column non-zero vector $\boldsymbol{v}_m$ and a scalar λ such that:

$$A\boldsymbol{v} = \lambda \boldsymbol{v}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

- λ is an **eigenvalue** of A
- $\boldsymbol{v}$ is an **eigenvector** associated with A

Eigenvalues & eigenvectors

Characteristic equation

- Transform above equation to:

$$(A - \lambda \boldsymbol{I})\boldsymbol{v} = \mathbf{0}$$

- A non-zero solution for $\boldsymbol{v}$ exists $\iff$ $det(A - \lambda \boldsymbol{I}) = 0$.

$$det\left(\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix}\right) = 0$$

- This equation is called characteristic equation of matrix A

Eigenvalues & eigenvectors

Characteristics

- Eigenvalues has some important characteristics:
  - Number of eigenvalues is always equal to the size of the matrix (there could be double eigenvalues)
  - $tr(A) = \sum_{i=1}^{m} \lambda_i$
  - $det(A) = \prod_{i=1}^{m} \lambda_i$
  - A is *invertible* if and only if every eigenvalue is non-zero
  - if A is invertible then eigenvalues of A^{-1} are equal $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_m}$