Brush-up Math Course

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Lecture Outline

1. Introduction - basic math concepts
2. Linear algebra
3. **Calculus**
4. Optimisation

Lecture 3: Calculus

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Lecture Outline

1. Limits
2. Continuity
3. Derivative
4. Implicit function
5. Integrals
6. Power series

Limits

Intuition

- Sometimes it is not possible to determine value of a function at some points, e.g.:
  - $f(x)|_{x=-2} = \frac{x^2-4}{x+2}|_{x=-2} = \frac{0}{0} = ?$
  - $f(x)|_{x=\infty} = \frac{1}{x}|_{x=\infty} = \frac{1}{\infty} = ?$
- So, instead of determining the value of the function at this point, we try to analyse how the function behaves if we are getting closer to this point
- e.g. $\lim\limits_{x\to -2} \frac{x^2-4}{x+2} = \lim\limits_{x\to -2} \frac{(x-2)(x+2)}{x+2} = \lim\limits_{x\to -2} (x-2) = -4$

Limits

Right/left limits

- The behaviour of a function around the point a can be analysed from:
  - the left $\lim_{x \to a^-} f(x)$
  - the right $\lim_{x \to a^+} f(x)$
- Function $f(x)$ has the limit at the point a if:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \neq \pm\infty$$

Limits

Right/left limits - example

- Let's consider $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x}$
  - from the right: $\lim_{x \to 0^+} \frac{1}{x} = \frac{1}{\frac{1}{\infty}} = \infty$
  - from the left: $\lim_{x \to 0^-} \frac{1}{x} = \frac{1}{\frac{1}{-\infty}} = -\infty$
- As a result:

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

- It means, that the limit of the function at the point 0, does not exist

Limits

Properties

- Assume that $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$
- Then:

$$\begin{aligned}
\lim_{x \to a} f(x) + g(x) &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) &&= A + B \\
\lim_{x \to a} f(x) - g(x) &= \lim_{x \to a} f(x) - \lim_{x \to a} g(x) &&= A - B \\
\lim_{x \to a} f(x) \cdot g(x) &= \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) &&= A \cdot B \\
\lim_{x \to a} f(x)/g(x) &= \lim_{x \to a} f(x)/\lim_{x \to a} g(x) &&= A/B \\
\lim_{x \to a} k f(x) &= k \lim_{x \to a} f(x) &&= k \cdot A
\end{aligned}$$

Continuity

Intuition

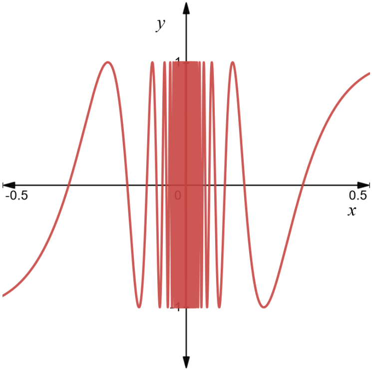
- *Continuous* - happening or existing for a some period without break
- *Continuous function* - function, which one can draw without lifting the pencil
- Formal definition: $f(x)$ with domain X is said to be continuous around the point $a \in X$ if:

$$f(a) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = A \neq \pm\infty$$

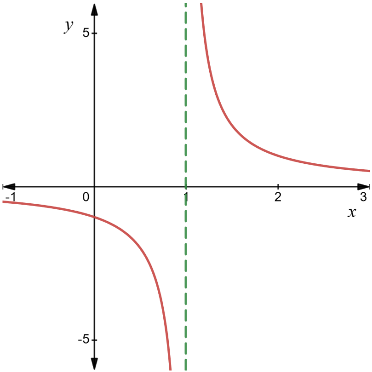
- Function $f(x)$ is called continuous, if it is continuous for every point in the domain

Discontinuity

Example



$f(x) = \sin\left(\frac{1}{x}\right)$



$f(x) = \frac{x-1}{x^2-2x+1}$

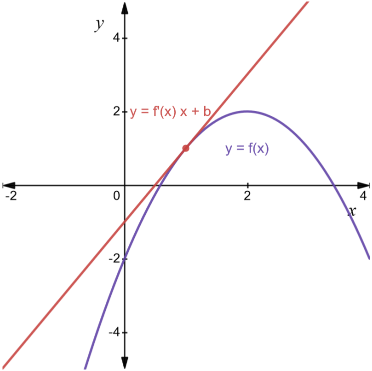
Derivative

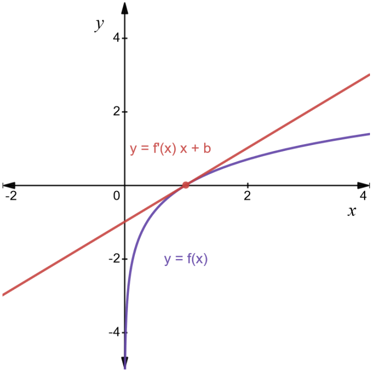
Geometric intuition

- Recall linear function: $f(x) = ax + b$
- The slope of the function is measured by a:
  - if a is positive, then the line rises from left to right
  - if a is negative, then the line falls
- But what about other functions? Polynomial? $\ln(x)$? e^x?
- Natural answer: define the slope of a curve at a particular point as the slope of the tangent to the curve at that point
- The slope of the tangent to the graph at given point is called *derivative* at that point

Derivative

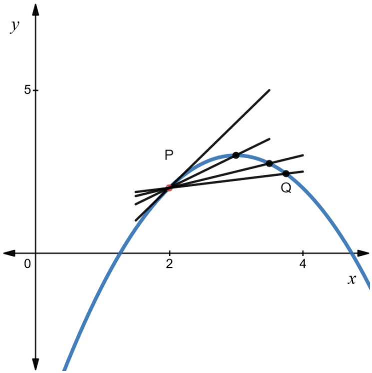
Geometric example





Derivative

Definition



- $P = (x, f(x))$
- $Q = (x+h, f(x+h))$
- Slope of the line PQ:
  $\frac{y_Q - y_P}{x_Q - x_P} = \frac{f(x+h) - f(x)}{h}$
- To find tangent:
  $Q \to P \implies h \to 0$
- Derivative of $f(x)$ at x:
  $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Derivative

Definition

- The equation for the tangent to the graph of $y = f(x)$ at the point $(a, f(a))$ is:

$$y = f'(a)(x - a) + f(a)$$

- A derivative of any function can by always using definition

- Example: find derivative of $f(x) = x^2 + 5x - 2$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 5x + 5h - 2 - x^2 - 5x + 2}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 2xh + 5h}{h} = \lim_{h \to 0} h + 2x + 5 = 2x + 5 \end{aligned}$$

Derivative

Existence

- Notice, that derivative exists $\Leftrightarrow$ limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists
- There are functions which are not differentiable on the whole domain or at some points
- The most trivial example is a module function:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1 \neq \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{h}{h} = 1$$

- Function to be differentiable has to be continuous at that point

Derivative

Notation

- Derivative are denoted in the literature in the several ways:
  - $f'(x)$
  - y'
  - $\frac{dy}{dx} (= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x})$
  - $\frac{d}{dx} f(x)$
  - $D_x(f(x))$
  - $\left.\frac{dy}{dx}\right|_{x=x_0}$ (at given point)
  - $\dot{y}$ - with respect to time t

Derivative

Common function

- Derivative can be always calculated using the definition
- However, it is useful to remember a few rules for basic functions
- Assume that a is a constant, then:
  - $\frac{d}{dx}a = 0$
  - $\frac{d}{dx}x^a = ax^{a-1}$ for $a \in \mathbb{R} \setminus \{0\}$
  - $\frac{d}{dx}e^x = e^x$
  - $\frac{d}{dx}a^x = a^x \ln a$
  - $\frac{d}{dx}\ln x = \frac{1}{x}$

Derivative

Combinations

- Set of rules to calculate derivative of combination of two or more functions:
  - $\frac{d}{dx}cf(x) = c \cdot \frac{d}{dx}f(x)$
    - $\frac{d}{dx}2\ln x = 2\frac{d}{dx}\ln x = \frac{2}{x}$
  - $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
    - $\frac{d}{dx}(e^x + x^2) = \frac{d}{dx}e^x + \frac{d}{dx}x^2 = e^x + 2x$

- continuation:
  - $\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
    - $\frac{d}{dx}(x \cdot e^x) = x \cdot \frac{d}{dx}e^x + e^x \frac{d}{dx}x = x \cdot e^x + e^x \cdot 1 = e^x(x+1)$
  - $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f(x)\frac{d}{dx}g(x) - g(x)\frac{d}{dx}f(x)}{(g(x))^2})$
    - $\frac{d}{dx}\frac{x}{\ln x} = \frac{\ln x \frac{d}{dx}x - x\frac{d}{dx}\ln x}{(\ln x)^2}) = \frac{\ln x - 1}{(\ln x)^2}$
  - $\frac{d}{dx}f(g(x)) = \frac{d}{dx}f(g(x)) \cdot \frac{d}{dx}g(x)$
    - $\frac{d}{dx}(e^{x^2}) = \frac{d}{dx}e^{x^2} \cdot \frac{d}{dx}x^2 = e^{x^2} \cdot 2x$

Derivative

Application - economics

- Derivatives have a lot of applications in economics
- The main one: rate of change - how much will change one variable with respect to the other
- Very often a word *marginal* refers to derivative, e.g.:
  - *marginal productivity of labour* = derivative of a production function w.r.t. labour - how the production changes if labour input increases by one unit
  - *marginal cost* = derivative of a cost function w.r.t. input - how the production cost changes if input increases by one unit
  - *marginal utility* = derivative of a utility function w.r.t. good - how the utility changes if amount of good increases by one unit

Derivative

Application - monotonicity

- Let's take function f and $\forall (a,b) \in I \subseteq D_f$ st. $a < b$, then f on an interval I is:
  - increasing (strictly) if $f(a) \leq f(b)$ $(f(a) < f(b))$
  - decreasing (strictly) if $f(a) \geq f(b)$ $(f(a) > f(b))$
  - constant if $f(a) = f(b)$
- Monotonicity can be concluded from the graph, but also using derivatives:
  - $f'(x) \leq 0$ for all $x \in I \Leftrightarrow f$ is decreasing in I
  - $f'(x) \geq 0$ for all $x \in I \Leftrightarrow f$ is increasing in I

Derivative

Application - extrema

- Extremum - the largest and smallest value of the function in the interval/domain

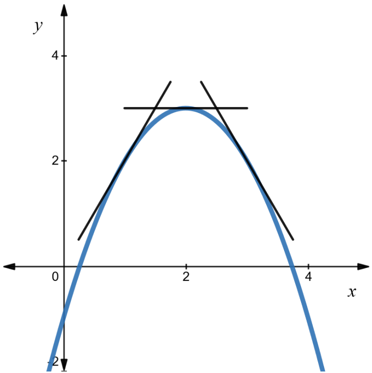
- If $f'(x_0) = 0$, then x_0 (*critical point*) can be:
  - global/local minimum
  - global/local maximum
  - inflection point

- If x_0 is an extremum and $f(x)$ is differentiable at x_0, then $f'(x_0) = 0$

- To determine if x_0 is an extremum we need higher order derivatives (see next slides)

Derivative

Application - monotonicity & extrema



- Slope of tangent is positive $\Leftrightarrow f'(x) > 0 \Leftrightarrow f(x)$ increases
- Slope of tangent is negative $\Leftrightarrow f'(x) < 0 \Leftrightarrow f(x)$ decreases
- Slope of tangent is zero $\Leftrightarrow f'(x) = 0 \Leftrightarrow ? f(x)$ has extremum (not always!)

Derivative

Higher order derivatives

- So far, we only discussed first order derivatives, but one can differentiate the function more than once
- N-order derivative of a function f is given by:

$$f^{n}(x) = (f^{(n-1)}(x))'$$

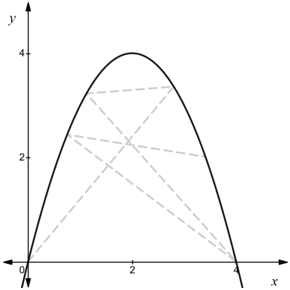
- So, second order derivative is

$$f^{(2)}(x) = f''(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2} = (f'(x))'$$

- Second order derivative is used to determine concavity and extrema of the function

Derivative

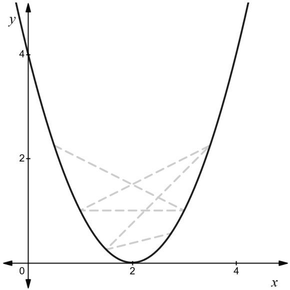
Concavity



- Intuition: function is concave if all lines graph of the function are below this graph
- Formally: $f(x)$ is concave if $\forall a, b \in I$, and $\forall \lambda \in [0,1]$ we have: $f((1-\lambda)a + \lambda b) \geq (1-\lambda)f(a) + \lambda f(b)$
- Using derivative: twice-differentiable $f(x)$ defined on I is concave if $\forall x \in I \;\; f''(x) \leq 0$

Derivative

Convexity



- Intuition: function is convex if all lines graph of the function are above this graph
- Formally: $f(x)$ is convex if $\forall a, b \in I$, and $\forall \lambda \in [0,1]$ we have: $f((1-\lambda)a + \lambda b) \leq (1-\lambda)f(a) + \lambda f(b)$
- Using derivative: twice-differentiable $f(x)$ defined on I is convex if $\forall x \in I \;\; f''(x) \geq 0$

Derivative

Second order derivative test

- Assume that $f(x)$ is twice-differentiable
- Then, assume that x_0 is a *critical point* ($f'(x_0) = 0$)
- If:
  - $f''(x_0) > 0$, then x_0 is a global/local minimum
  - $f''(x_0) < 0$, then x_0 is a global/local maximum
  - $f''(x_0) = 0$, then the test is inconclusive (x_0 can be max, min or an inflection point)

Multivariate calculus

Introduction

- So far, we focus only on the function of one variable, but function can be of more variables, e.g.:

$$z = f(x, y)$$

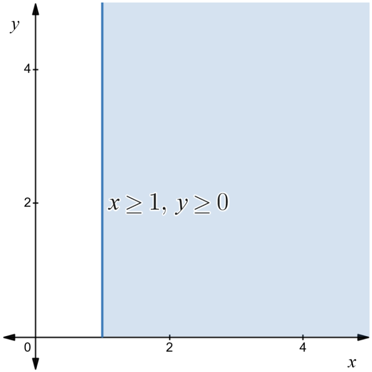
depends on two variables: x and y.

- Function might have any number of arguments:

$$z = f(\boldsymbol{x}) = f(x_1, x_2, x_3, ..., x_n)$$

- In case of functions with multiple arguments, we can defined them as a vector:

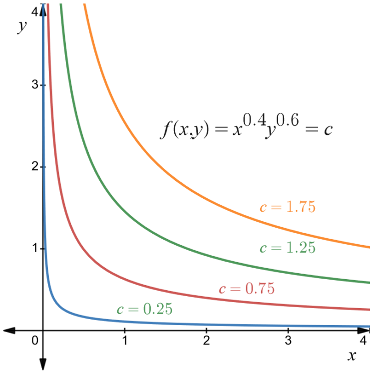
$$\boldsymbol{x} = (x_1, x_2, x_3, .., x_n)$$



- Before conducting further analysis, it is important to define the domain of a function
- e.g.:
  $f(x, y) = \sqrt{x-1} + \sqrt{y}$,
  $D = [1, \infty) \times [0, \infty)$

Multivariate calculus

Level curves



- *Level curve* - the line over the xy-plane where the function takes the same value
- Formally: $z = f(x, y) = c$
- The collection of level curves is called the *contour-map*

Multivariate calculus

Level curves - economics

- Level curves are very important in economics, e.g.:
  - If $f(L, K)$ is a production function, then the level curve represents combination of two inputs which results in the same amount of output
  - If $C(q_0, q_1)$ is a cost function, then the level curve represents combination of two production inputs which results in the same cost level
  - If $u(x_0, x_1)$ is a utility function, then the level curve represents combination of two goods which results in the same level of utility

Multivariate calculus

Partial derivatives

- *Partial derivative* - the derivative of a multivariate function w.r.t. one of its variables
- Partial derivative measures how the value of the function changes, if one variable changes while the rest is kept constant
- Consider $f(x, y)$, then partial derivatives are as follows:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x(x, y)$$
$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} = f_y(x, y)$$

Multivariate calculus

Partial derivatives

- While calculating partial derivatives, we can use the same rules like for single-variable case
- If we calculate derivative w.r.t. one variable, then we treat all other variables as constants
- e.g.:

$$f(x,y) = x^4 + 3x^2y^3 - \ln\left(2x^2y\right)$$

$$f_x = 4x^3 + 6xy^3 - \frac{2}{x}$$

$$f_y = 9x^2y^2 - \frac{1}{y}$$

Multivariate calculus

Gradient

- Very often, one needs to calculate partial derivatives w.r.t. all function's arguments
- We can group all partial derivatives into a vector called *gradient*:

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\boldsymbol{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\boldsymbol{x}) \end{bmatrix}$$

- In the example from the previous slide, the gradient is:

$$\nabla f(x, y) = \begin{bmatrix} 4x^3 + 6xy^3 - \frac{2}{x} \\ 9x^2y^2 - \frac{1}{y} \end{bmatrix}$$

Multivariate calculus

Higher order partial derivatives

- We can also calculate the *higher order partial derivatives* for multivariate functions
- The procedure is similar to the one for the first order derivatives
- Let's take function of two variables $f(x, y)$, then *second order partial derivatives* are:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

- In our example first order derivatives are:

$$\nabla f(x,y) = \begin{bmatrix} 4x^3 + 6xy^3 - \frac{2}{x} \\ 9x^2y^2 - \frac{1}{y} \end{bmatrix}$$

- Then second order derivatives are:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 12x^2 + 6y^3 + \frac{2}{x^2} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 18xy^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 18xy^2 \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 18x^2y + \frac{1}{y^2}$$

- Notice, that $f_{xy} = f_{yx}$, which is true if the second partial derivatives are continuous

Multivariate calculus

Hessian

- Second order derivatives can be grouped into symmetric (!) matrix called *Hessian*:

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- In our example, the Hessian is:

$$H_f(x, y) = \begin{bmatrix} 12x^2 + 6y^3 + \frac{2}{x^2} & 18xy^2 \\ 18xy^2 & 18x^2 y + \frac{1}{y^2} \end{bmatrix}$$

Implicit function

Introduction

- So far, we focused on the explicit formulas like $y = f(x)$
- Now, let's focus on function defined implicitly by an equation such as $g(x, y) = c$, where c is a constant
- Consider the following examples:

$$g(x, y) = xy = 5$$
$$h(x, y) = y^3 + 3x^2y = 13$$

- This form of defining a function is used e.g. to find level curves

Implicit function

Chain rule

- So far, we know a chain rule for one-variable function:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

- Assume that f is a following multivariate function:

$$f(g_1(x), ..., g_n(x))$$

- Then the following chain rule can be implemented:

$$\frac{d}{dx} f(g_1(x), ..., g_n(x)) = \sum_{i=1}^{n} \left(\frac{\partial}{\partial g_i(x)} f(g_1(x), ..., g_n(x)) \right) \left(\frac{d}{dx} g_i(x) \right)$$

Implicit function

Implicit differentiation

- We can differentiate an implicit function using a chain rule
- First, define $y = f(x)$. Then $g(x, y) = g(x, f(x))$
- Then differentiate $g(x, f(x)) = c$ w.r.t. x using the chain rule

$$\frac{\partial g(x, f(x))}{\partial x} = \frac{\partial g(x, y)}{\partial x}\frac{dx}{dx} + \frac{\partial g(x, y)}{\partial y}\frac{dy}{dx} = 0$$

- Solving for $\frac{dy}{dx}$ gives us:

$$\frac{dy}{dx} = -\frac{\frac{\partial g(x,y)}{\partial x}}{\frac{\partial g(x,y)}{\partial y}} = -\frac{g_x(x, y)}{g_y(x, y)}$$

Implicit function

Economic example

- Assume that $F(L,K) = K^{\alpha}L^{1-\alpha}$ is a production function with two inputs: L - labour and K - capital
- Then, assume that we want to produce the amount of a good equal to c: $F(L,K) = c$
- Question: if we increase labour by one unit, how do we need to adjust capital to keep the same level of production?

$$\frac{dK}{dL} = -\frac{F_K}{F_L} = -\frac{\alpha K^{\alpha-1}L^{1-\alpha}}{(1-\alpha)K^{\alpha}L^{-\alpha}} = -\frac{\alpha}{(1-\alpha)}\frac{L}{K}$$

Integrals

Intuition

- So far, we focus on finding the slope of a function. We also analyse its monotonicity and extrema
- Now, let's invert the problem: if we know the derivative of the function, can we recover the original function?
- Yes, if we use *indefinite integration*!
- Another interpretation of integration - *definite integration* - is a concept of calculating the area under the graph of a function
- Integrals appear in economics when we analyse consumer surplus, lifetime utility or income distribution.

Integrals

Indefinite integration

- *Indefinite integration* operation is denoted by:

$$F(x) = \int f(x)dx$$

where:

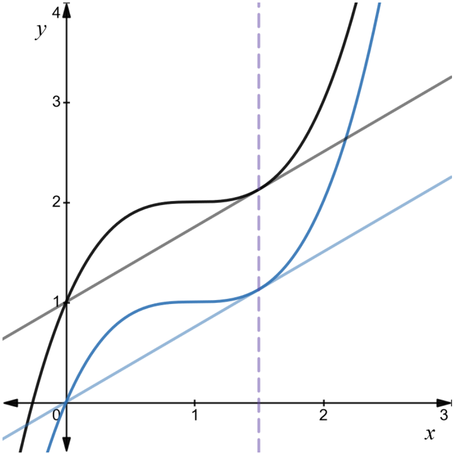
  - $\int$ - an integration operator
  - $f(x)$ - which function we integrate
  - dx - along which variable we integrate this function

- *Indefinite integration* inverts differentiation, formally:

$$F'(x) = f(x) \Leftrightarrow F(x) = \int f(x)dx$$

Integrals

Indefinite integration



$f(x) = 3(x-1)^2 \Leftrightarrow F(x) = (x-1)^3 + C$

- Notice that: $(f(x) + C)' = f'(x)$
- So, when we integrate, we have to recover C:

$$\int f(x)dx = $$
$$= F(x) + C$$

Integrals

Common function

- Integrating is not trivial, so it is useful to remember a few rules for basic functions
- Assume that a is a constant, then:
  - $\int a\ dx = ax + C$
  - $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
  - $\int \frac{1}{x} dx = \ln|x| + C$
  - $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$
  - $\int \ln x\ dx = x \ln x - x + C$

Integrals
Combinations

- Set of rules to integrate a combination of two or more functions:
  - $\int cf(x)dx = c\int f(x)dx$
    - $\int 2x^2\,dx = 2\int x^2\,dx = 2\cdot\frac{x^3}{3} = \frac{2}{3}x^3$
  - $\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$
    - $\int (e^x + x)\,dx = \int e^x dx + \int x\,dx = e^x + \frac{x^2}{2}$
- Notice, that in contrary to differentiation, there is no rule for an integral of multiplication or division of two functions

Integrals

Substitution

- To compute integral using substitution method, first we have to be able to present it in the following way:

$$\int f(g(x))g'(x)dx$$

- e.g.: $\int 2xe^{x^2}dx$

- Then, let's define u, such that:

$$u = g(x)$$
$$du = g'(x)dx$$

- in our example: $u = x^2$ and $du = 2x\ dx$

- Then, we need to substitute for u and du in the integral
- In our example:

$$\int e^u \, du = e^u + C$$

- Now, substitute back for u:

$$e^u + C = e^{x^2} + C$$

Integrals

Substitution

- Unfortunately, it is not always trivial (and possible) to present the integral in the form which allows us to use substitution, e.g.:

$$\int \frac{x}{x^2+1} dx$$

- It sometimes requires reorganisation of the equation to make it possible, i.e:

$$\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx$$

- Then: $u = (x^2 + 1)$ and $du = 2xdx$

Integrals

Integration by parts

- However, there are a lot of integrals which cannot be computed using substitution method
- Some of them we can compute using integration by parts
- Recall:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

- Rearranging and integrating both sides gives us:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Integrals

Integration by parts

- Intuition: we divide our integral into two function: $f(x)$ (easy to differentiate) and $g'(x)$ (easy to integrate)
- Example:

$$\int \frac{\ln x}{x^2} dx$$

- Then:

$$f(x) = \ln x \quad f'(x) = \frac{1}{x}$$

$$g(x) = \frac{-1}{x} \quad g'(x) = \frac{1}{x^2}$$

- Now, let's substitute it into the integral:

$$\int \frac{\ln x}{x^2}dx = \ln x \cdot \frac{-1}{x} - \int \frac{1}{x}\cdot\frac{1}{x}dx$$
$$= \frac{-\ln x}{x} - \frac{1}{x} + C = -\frac{\ln x + 1}{x} + C$$

- Unfortunately, it is often not trivial to rewrite integral as $f(x)$ and $g'(x)$, e.g.: $\int \ln x \, dx = \int 1 \cdot \ln x \, dx$

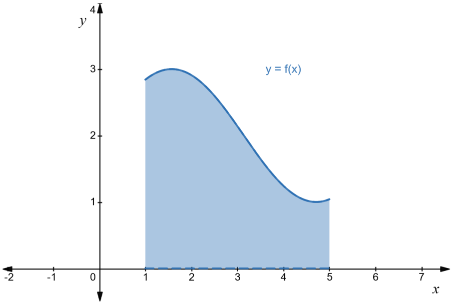
- Then:

$$f(x) = \ln x \quad f'(x) = \tfrac{1}{x}$$
$$g(x) = x \qquad g'(x) = 1$$

Integrals

Definite integrals

- So far we focus only on the first idea of integration as an inverse of differentiation
- Now, let's focus on the second idea - computing the area under the graph
- To do it, we are going to use *definite integrals*



Integrals

Definite integrals

- Let $f(x)$ be a continuous non-negative function in the close interval $[a, b]$
- Then the definite integral is:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where:

- $\int$ - integration operator
- a, b - downer and upper limits of integration
- $f(x)$ - function to be integrated
- dx - along which variable the function is integrated
- $F(x)$ - indefinite integral of $f(x)$

Integrals

Definite integrals

- To calculate a definite integral for some interval we need:
  - Ensure that the function is non-negative on the given interval
  - Calculate indefinite integral
  - Use formula from the previous slide
- For definite integrals we have additional rules:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Integrals

Definite integrals - example

- Let's take function $f(x) = \frac{1}{x}$ and calculate its definite integral on the close interval $[1, e]$

$$\int_1^e \frac{1}{x} = \ln x]_1^e = \ln e - \ln 1 = 1 - 0 = 1$$

- Now, let's take function $f(x) = -x^2$ and calculate its definite integral on the close interval $[-1, 1]$
- However the function has negative values on this interval
- Then the outcome of the integration will be minus area above the graph on this interval

Power series

Definition

- Power series are functions of the following form:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

- Coefficients a_n are constants
- It has a form of infinite sums
- Power series are continuous, differentiable and integrable

Power series

Taylor series

- Is it possible to expressed any differentiable function in terms of a power series?
- It turns out that it is possible to do so within the radius of convergence.
- It means, that power series could help us to approximate other function around a point
- Formally:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- You can check an intuition for $\ln(1+x)$ here.

Power series

Taylor series - example

- Let's find the Taylor series for $f(x) = \ln x$ about $x = 2$

$$
\begin{aligned}
f^{(0)}(x) &= \ln(x) & \quad f^{(0)}(2) &= \ln 2 \\
f^{(1)}(x) &= \tfrac{1}{x} & \quad f^{(1)}(2) &= \tfrac{1}{2} \\
f^{(2)}(x) &= -\tfrac{1}{x^2} & \quad f^{(2)}(2) &= -\tfrac{1}{2^2} \\
f^{(3)}(x) &= \tfrac{2}{x^3} & \quad f^{(3)}(2) &= \tfrac{2}{2^3} \\
&\vdots \\
f^{(n)}(x) &= \tfrac{(-1)^{n+1}(n-1)!}{x^n} & \quad f^{(n)}(2) &= \tfrac{(-1)^{n+1}(n-1)!}{2^n}
\end{aligned}
$$

- Substituting into Taylor series formula:

$$\ln x = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(x-2)^n$$