Brush-up Math Course

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Lecture Outline

1. Introduction - basic math concepts
2. Linear algebra
3. Calculus
4. **Optimisation**

Lecture 4: Optimisation

September 2021

Lecture Outline

1. Introduction
2. Unconstrained optimisation
3. Envelop theorem
4. Equality constraints
5. Inequality constraints

Introduction

Intuition

- Finding the best way to do a specific task involves what is called an *optimisation problem*.
- Studying an optimisation problem in the systematic way requires a mathematical model
- In economics, we assume that agents are endowed with a *payoff function*
- Payoff function maps the cross-product of players strategy spaces to the players set of payoffs
- e.g.: consuming 2 units of good 1 (strategy) gives me utility equal 2 (payoff)

Introduction

Intuition

- Assume form of payoff function and rational behaviour of agents
- Then we can start considering which strategies of agents will give her the highest payoff
- This leads us to the optimisation problem
- Optimisation can be unconstrained or constrained
- Examples:
  - Consumers are meant tomaximise their utility over purchases
  - Firms are supposed to maximise profits over investments
  - Parties maximise votes over programmes

Introduction

Definition

- Let $f(x)$ be a function with domain X and let S be a subset of a domain.
- The maximisation problem has the following form:

$$\max_{x} f(x)$$
$$\text{subject to } x \in S$$

- Solution to the problem is x^* s.t. $f(x) \leq f(x^*)\ \forall \mathbf{x} \in S$
- x^* is a maximiser of $f(x)$ and $f\left(x^*\right)$ is the maximum of $f(x)$ subject to the constraint $x \in S$

Introduction

Single-variable case

- During the previous lectures we were analysing extrema of single-variable function without constrain
- Unfortunately, the optimisation problems are usually more complicated
- We are going to analyse optimisation problems with:
  - many variables
  - many constrains in form of equality or inequality

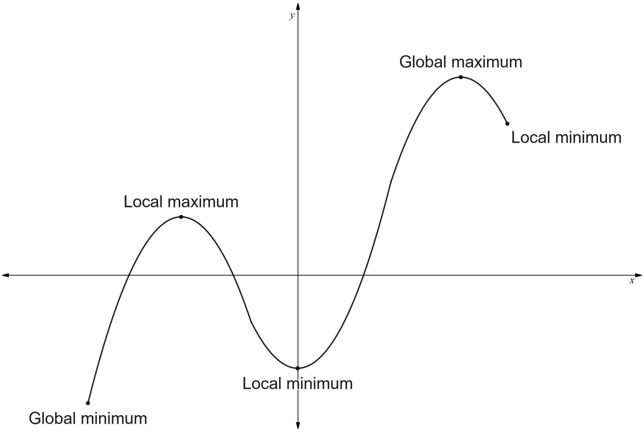
Introduction

Local vs global - definition

- Recall the difference between global and local extrema
- The point x^* is a *local maximum* of $f(x)$ subject to $x \in S$, if there is a number $\epsilon > 0$ s.t. $f(x) \leq f(x^*)$ for $\forall x \in [x^* - \epsilon, x^* - \epsilon]$
- The point x^* is a *global maximum* of $f(x)$ subject to $x \in S$, if $f(x) \leq f(x^*)$ for $\forall x \in X$

Introduction

Local vs global - graph



Global and local minima and maxima

Introduction

Monotonic transformation

- To simplify optimisation problem, one can use *monotonic transformation*
- Let's assume that we have a following maximisation problem:

$$\max_x f(x) \text{ s.t. } x \in S$$

- Then, take any strictly incising function $g(x)$ on S
- The following maximisation problem:

$$\max_x g(f(x)) \text{ s.t. } x \in S$$

has the same set of solution as the initial problem.

Introduction

Monotonic transformation - example

- The most common example of usage of monotonic transformation is Cobb-Douglas function
- Assume that firm faces following maximisation problem:

$$\max_{K\geq 0, L\geq 0} L^{1-\alpha}K^{\alpha}$$

- Let's transform the problem using ln() function

$$\max_{K\geq 0, L\geq 0} (1-\alpha)\ln(L)+\alpha\ln(K)$$

- The new problem is easier to work with, since it is linear w.s.t. parameter α

Introduction

Minimisation

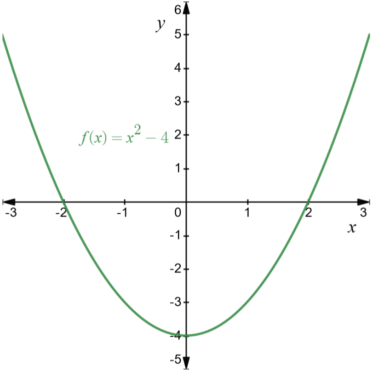
- So far, we only discuss about maximisation problem
- What happens if we want to e.g. minimise cost function?
- Minimisation problem is an equivalent to maximise minus problem

$$\min_{x} f(x) \quad \text{is equal to} \quad \max_{x} -f(x)$$
$$\text{s.t. } \mathbf{x} \in S \qquad\qquad \text{s.t. } x \in S$$

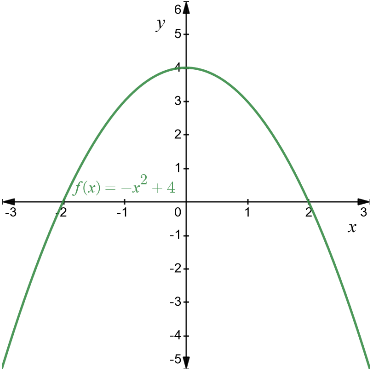
- Because of that, I will focus only on showing how to solve maximisation problem

Introduction

Minimisation - example



$\min_x f(x)$

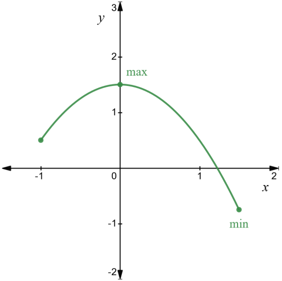


$\max_x -f(x)$

Introduction

Extreme value theorem

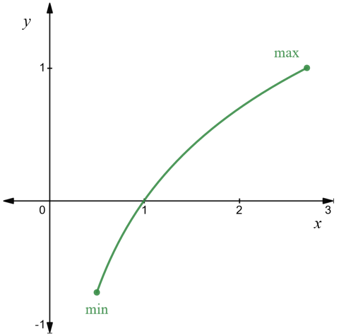
- Suppose that f is a *continuous* function over a compact (*closed* and *bounded*) interval $[a, b]$.
- Then there exists a point d in $[a, b]$ where f has a minimum and a point c in $[a, b]$ where f has a maximum.
- That is, one has $f(d) \leq f(x) \leq f(c)$ for all x in $[a, b]$.



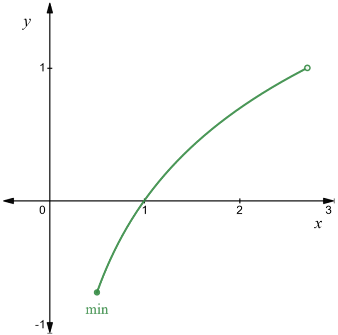
Introduction

Closedness

- The interval is *closed* if the interval includes all its limit points i.e.: $[0,1]$, $(-\infty,0]$, $(-\infty,\infty)$
- What happens if the set is not closed?



Closed

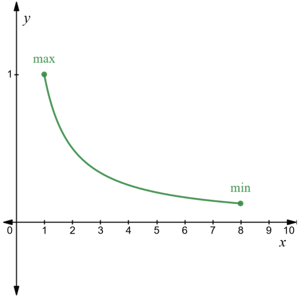


Open

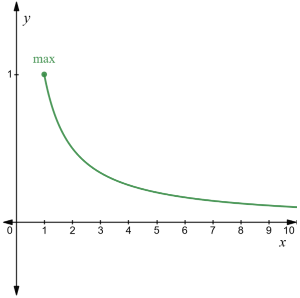
Introduction

Boundness

- The interval is *bounded* if there are two real numbers that are, respectively, smaller than and larger than all its elements
- What happens if the set is not bounded?



Bounded

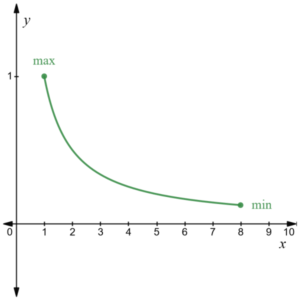


Unbounded

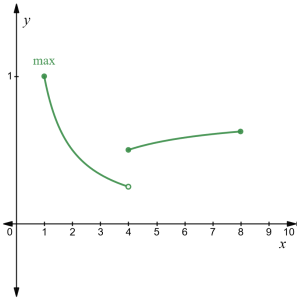
- The function is *continuous* if:

$$f(a) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = A \neq \pm\infty$$

- What happens if the function is discontinuous?



Continuous



Discontinuous

Unconstrained optimisation

Introduction

- First, we will focus on the case, when we solve an optimisation problem without the constraint
- The general formula for n-dimensional maximisation problem is:

$$\max_x f(x)$$
$$\text{s.t. } x \in X$$

  where X is a domain of a $f(x)$

- Unconstrained $\longrightarrow$ we are looking for extrema in the whole domain

Unconstrained optimisation

Example

- Consider that we would like to solve the following maximisation problem:

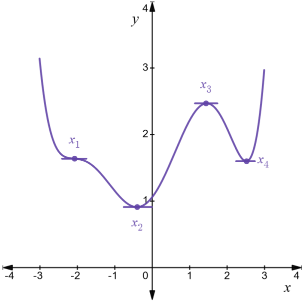
$$\max_{x,y} f(x,y) = \max_{x,y} \exp(-\frac{1}{3}x^3 + x - y^2)$$

- Unconstrained optimisation: $(x, y) \in \mathbb{R}^2$
- The problem does not meet the conditions of extreme value theorem (unbounded), so we do not know if it has extrema

Unconstrained optimisation

Stationary point

- Let's assume, that the function $f(x)$ is defined on a domain X. A point $x' \in X$ is called *stationary point* of $f(x)$ if $f(x')$ is differentiable and $f'(x) = 0$



- x_1, x_2, x_3, x_4 are stationary points, however x_1 is not an extremum

Unconstrained optimisation

Interior solution

- To sum up:
  - A stationary point doesn't have to be a local extremum
  - A local extremum might not be a stationary point
- Notice, that:
  - A local extremum is not a stationary point only if it lies at the boundary of the set
  - So, if an interior point is a local extremum, then it has to be a stationary point too
- It means that being a stationary point is a necessary but not sufficient condition for a point to be a local extremum

First-order condition (FOC)

Let's assume, that the function $f(x)$ is defined on a domain X, x^* is a interior point, which is extremum of the function and $f'(x^*)$ exists. Then, $f'(x^*) = 0$ (or in other words x^* is a stationary point of $f(x)$).

- It means, that stationary points are only "suspected" to be an extremum
- Assuming that a stationary point is an extremum, we still do not know if it is a minimum or maximum

Unconstrained optimisation

Hessian

- Recall Hessian from the previous lecture:

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- It is a symmetric matrix ($f_{ij} = f_{ji}$)
- We are going to use Hessian to determine if the stationary points are local maxima/minima

Unconstrained optimisation

Definite matrix

- The Hessian (H), as every symmetric squared matrix, can be definite:
  - H positive-definite $\Leftrightarrow x^\top Hx > 0$ for all $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$
  - H positive semi-definite $\Leftrightarrow x^\top Hx \geq 0$ for all $x \in \mathbb{R}^n$
  - H negative-definite $\Leftrightarrow x^\top Hx < 0$ for all $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$
  - H negative semi-definite $\Leftrightarrow x^\top Hx \leq 0$ for all $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$
  - H is indefinite $\Leftrightarrow$ H is neither positive semi-definite nor negative semi-definite

Unconstrained optimisation

Definite matrix

- e.g. I_2 positive definite:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 > 0 \text{ for all } x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$$

- For M_{2x2} matrix we can use the following formula:
  - If $det(M_{2x2}) > 0$ and $M_{0,0} > 0 \implies$ M is positive definite
  - If $det(M_{2x2}) > 0$ and $M_{0,0} < 0 \implies$ M is negative definite
  - If $det(M_{2x2}) < 0 \implies$ M is indefinite

Second-order condition (SOC)

Let's assume, that the function $f(x)$ is defined on a domain X and x^* is a stationary point of $f(x)$. Then, assume that $f(x)$ is twice-differentiable and $H(x)$ is its Hessian. Then, x^* is a:

- a local minimum if $H(x^*)$ is positive definite
- a local maximum if $H(x^*)$ is negative definite
- a saddle point if $H(x^*)$ is indefinite

Test is inconclusive, if $det(H(x^*)) = 0$.

Unconstrained optimisation

Procedure

Procedure to solve an unconstrained optimisation problem

- Assume that $f(x)$ is a differentiable function of n-variables define on a domain X
- Write the maximisation problem:

$$\max_{x} f(x)$$
$$\text{s.t. } x \in X$$

- Assuming that the solution exists, it can be found in the following way:
  1. Use the FOC to find all stationary points
  2. Use the SOC to determine if stationary points are max/min/saddle point

Unconstrained optimisation

Example 1

- Solve the following problem:

$$\max_{x,y} f(x,y) = \max_{x,y} -(x-1)^2 - (y-2)^2$$

- We do not know in front if the problem has a solution
- FOCs:

$$\begin{aligned} f_x(x,y) &= -2(x-1) = 0 \implies x = 1 \\ f_y(x,y) &= -2(y-2) = 0 \implies y = 2 \end{aligned}$$

- Stationary point (x,y): $(1,2)$

Unconstrained optimisation

Example 1 - continuation

- Hessian of the function is given by:

$$H(x,y) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = H$$

- $det(H) = 4 > 0$
- $H_{0,0} = -2 < 0$
- It means that $f(x,y)$ has a local maximum at $(1,2)$

Unconstrained optimisation

Example 2

- Solve the following problem:

$$\max_{x,y} f(x,y) = \max_{x,y}(x-2)^3 + (y+2)^3$$

- We do not know in front if the problem has a solution

- FOCs:

$$f_x(x,y) = 3(x-2)^2 = 0 \implies x = 2$$
$$f_y(x,y) = 3(y+2)^2 = 0 \implies y = -2$$

- Stationary point (x,y): $(2,-2)$

Unconstrained optimisation

Example 2 - continuation

- Hessian of the function is given by:

$$H(x, y) = \begin{bmatrix} 6(x-2) & 0 \\ 0 & 6(y+2) \end{bmatrix} \implies H(2, -2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- $det(H(2, -2)) = 0 \implies$ SOC test is inconclusive

Unconstrained optimisation

Example 3

- Solve the following problem:

$$\max_{x,y} f(x,y) = \max_{x,y} x^3 + y^3 - 6xy$$

- We do not know in front if the problem has a solution
- FOCs:

$$\begin{cases} f_x(x,y) & = 3x^2 - 6y = 0 \\ f_y(x,y) & = 3y^2 - 6x = 0 \end{cases}$$

- Stationary points (x,y): $(0,0),(2,2)$

Unconstrained optimisation

Example 3 - continuation

- Hessian of the function is given by:

$$H(x,y) = \begin{bmatrix} 6x & -6 \\ -6 & 6y \end{bmatrix} \quad H(0,0) = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix} \quad H(2,2) = \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix}$$

- $det(H(0,0)) = -12 \implies (0,0)$ is a saddle point
- $det(H(2,2)) > 0$ and $H_{0,0} > 0 \implies (2,2)$ is a local minimum

Envelop theorem

Introduction

- In economic theory we are often interested in how the maximal value of a function depends on some parameters

- Assume that we would like to maximise the following function:

$$\max_x f(x; \boldsymbol{r}) = \max_x x^{r_1} - r_2 x$$
$$\text{where } x \geq 0 \text{ and } r_1 \in (0,1)$$

- x is a variable and $\boldsymbol{r}$ is a vector of parameters

- Condition on parameters value solution to this problem is:

$$x^*(\boldsymbol{r}) = \left(\frac{r_1}{r_2}\right)^{\frac{1}{1-r_1}}$$

Envelop theorem

Introduction

- Value of the function at maximum is:

$$\max_{x} f(x;\boldsymbol{r}) = f^*(\boldsymbol{r}) = \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{1-r_1}} - r_2\left(\frac{r_1}{r_2}\right)^{\frac{1}{1-r_1}}$$

- How the function at the solution $x^*(\boldsymbol{r})$ changes as some parameters $\boldsymbol{r}$ change?

- Using chain rule:

$$\frac{\partial f^*(\boldsymbol{r})}{\partial r_k} = \frac{\partial f\left(x^*(\mathbf{r});\mathbf{r}\right)}{\partial x^*(\mathbf{r})} \cdot \frac{\partial x^*(\mathbf{r})}{\partial r_k} + \frac{\partial f\left(x^*(\mathbf{r});\mathbf{r}\right)}{\partial r_k}$$

- But in optimum: $\frac{\partial f(x^*(\mathbf{r});\mathbf{r})}{\partial x^*(\mathbf{r})} = 0$, so $\frac{\partial f^*(\boldsymbol{r})}{\partial r_k} = \frac{\partial f(x^*(\mathbf{r});\mathbf{r})}{\partial r_k}$

Envelop theorem

Proposition

Envelope theorem for an unconstrained maximization problem

Let f be a function of $n+k$ variables, let $\boldsymbol{r}$ be a k-vector, and let the n-vector $\boldsymbol{x}^*$ be a maximizer of $f(\boldsymbol{x};\boldsymbol{r})$. Assume that the partial derivative of f with respect to r_h at $(\boldsymbol{x}^*,\boldsymbol{r})$ exists. Define the function f^* of k variables by:

$$f^*(\boldsymbol{r}) = \max_{x} f(\boldsymbol{x};\boldsymbol{r}) \text{ for all } \boldsymbol{r}$$

If the partial derivative $\frac{\partial f^*(\boldsymbol{r})}{\partial r_h}$ exists then:

$$\frac{\partial f^*(\boldsymbol{r})}{\partial r_h} = \frac{\partial f\left(x^*(\mathbf{r});\mathbf{r}\right)}{\partial r_h}$$

Envelop theorem

Example

- Thus by the envelope theorem, the derivative of the maximal value of f with respect to r_1 is the derivative of f with respect to r_1 evaluated at $x^*(r)$, namely:

$$\frac{\partial f\left(x^*(\mathbf{r});\mathbf{r}\right)}{\partial r_1} = \frac{\partial}{\partial r_1}\left((x^*(\mathbf{r}))^{r_1} - r_2 x^*(\mathbf{r})\right) = (x^*(\mathbf{r}))^{r_1} \ln x^*(\mathbf{r})$$

- Then, we can substitute for $x^*(\mathbf{r})$:

$$\frac{\partial f\left(x^*(\mathbf{r});\mathbf{r}\right)}{\partial r_1} = \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{1-r_1}} \frac{1}{1-r_1} \ln\left(\frac{r_1}{r_2}\right)$$

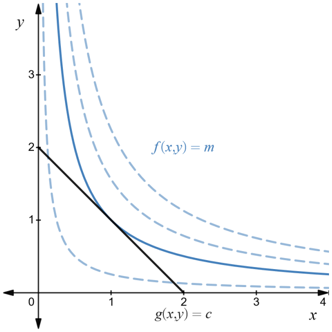
Constrained optimisation

Introduction

- So far, we analyse only unconstrained cases
- However, in economics we rather face a constrained optimisation e.g. budget constraint
- There are two types of constraints:
  - Equality constraint: $g(x) = c$
  - Inequality constraint: $g(x) \leq c$
- First, we focus on equality constraints

Equality constraints

Introduction



- Consider the constrained maximisation problem:

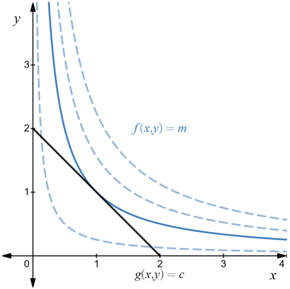
$$\max_{x,y} f(x,y)$$
$$\text{s.t. } g(x,y) = c$$

- The solution is the point (x^*, y^*) where both lines are tangent:

$$-\frac{f_x(x^*,y^*)}{f_y(x^*,y^*)} = -\frac{g_x(x^*,y^*)}{g_y(x^*,y^*)}$$

Equality constraints

Introduction



- Rearranging the equation, we get:

$$-\frac{f_x(x^*, y^*)}{g_x(x^*, y^*)} = -\frac{f_y(x^*, y^*)}{g_y(x^*, y^*)} = \lambda$$

- The set of condition to be met in equilibrium is:

$$\begin{aligned} f_x(x^*, y^*) - \lambda g_x(x^*, y^*) &= 0 \\ f_y(x^*, y^*) - \lambda g_y(x^*, y^*) &= 0 \\ c - g(x^*, y^*) &= 0 \end{aligned}$$

Equality constraints

Lagrangian

- Which function would generate such set of condition?
- Consider:

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$$

- Then, FOC for this function provide us a set of conditions equivalent to the constrained maximisation problem

Equality constraints

Necessary condition

Necessary condition

Let's assume, that $f(x, y)$ and $g(x, y)$ are continuously differentiable functions defined on domain X, c is a constant and (x^*, y^*) is the interior solution to the following maximisation problem:

$$\max_{x,y} f(x, y)$$
$$\text{s.t. } g(x, y) = c$$

Then, there exists a unique number λ^*, s.t. (λ^*, x^*, y^*) is a stationary point of Lagrangian:

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$$

Equality constraints

Second-order condition

- Using Lagrangian method, we can transform optimisation problem with equality constraint to unconstrained optimisation problem
- Similarly like in case of unconstrained optimisation, stationary points of Lagrangian function are only suspected to be an extrema of our original problem
- To decide if stationary point is an extremum, we need second-order condition

Equality constraints

Sufficient condition

Sufficient condition

Let's assume, that $f(x, y)$ and $g(x, y)$ are continuously twice-differentiable functions defined on domain X, c is a constant and (λ^*, x^*, y^*) is a stationary point of the Lagrangian function:

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$$

Assume that $H(\lambda^*, x^*, y^*)$ is a Hessian of a Lagrangian function, if:

- $det(H(\lambda^*, x^*, y^*)) > 0$, then (x^*, y^*) is a maximiser of the initial problem
- $det(H(\lambda^*, x^*, y^*)) < 0$, then (x^*, y^*) is a minimiser of the initial problem

Equality constraints

Sufficient condition

- Notice, that the Hessian of a Lagrangian function has the following form:

$$H(\lambda, x, y) = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda x} & L_{\lambda y} \\ L_{x\lambda} & L_{xx} & L_{xy} \\ L_{y\lambda} & L_{yx} & L_{yy} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & g_x & g_y \\ g_x & f_{xx} - \lambda g_{xx} & f_{xy} - \lambda g_{xy} \\ g_y & f_{xy} - \lambda g_{xy} & f_{yy} - \lambda g_{yy} \end{bmatrix}$$

Equality constraints

Procedure

Procedure to solve an two-variable optimisation problem with equality constraints

- Assume that $f(x, y)$ is a twice-differentiable function of define on a domain X
- Rewrite the maximisation problem using Lagrangian function:

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$$

- Assuming that the solution exists, it can be found in the following way:
  1. Use the FOC to find all stationary points of the Lagrangian function
  2. Use the SOC to determine if stationary points are max/min

Constrained optimisation - equality

Example 1

- Solve the following problem:

$$\max_{x,y} -(x-1)^2-(y-2)^2$$
$$\text{s.t. } x+y=2$$

- We do not know in front if the problem has a solution (set is not compact)

- Rewrite the problem using Lagrangian function:

$$\max_{\lambda,x,y} = -(x-1)^2-(y-2)^2-\lambda(x+y-2)$$

Constrained optimisation - equality

Example 1 - continuation

- FOCs of Lagrangian function:

$$\begin{aligned} L_\lambda &= -x - y + 2 = 0 \\ L_x &= -2(x-1) - \lambda = 0 \\ L_y &= -2(y-2) - \lambda = 0 \end{aligned}$$

- Stationary point is: $(\lambda^*, x^*, y^*) = (-1, \frac{1}{2}, \frac{3}{2})$

- The Hessian is:

$$H(\lambda, x, y) = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

- $det(H(\lambda, x, y)) = 4 > 0 \implies (\frac{1}{2}, \frac{3}{1})$ is a local maximum of the initial problem

Constrained optimisation - equality

Example 2

- Solve the following problem:

$$\max_{x,y} x^2(y-1)^2$$
$$\text{s.t. } x^2 + y^2 = 1$$

- We know in front if the problem has a solution (a unit circle is a compact set)

- Rewrite the problem using Lagrangian function:

$$\max_{\lambda,x,y} = x^2(y-1)^2 - \lambda(x^2 + y^2 - 1)$$

Constrained optimisation - equality

Example 2 - continuation

- FOCs of Lagrangian function:

$$\begin{aligned} L_\lambda &= -x^2 - y^2 + 1 = 0 \\ L_x &= 2x(y-1)^2 - 2\lambda x = 0 \\ L_y &= 2x^2(y-1) - 2\lambda y = 0 \end{aligned}$$

- Stationary points (λ^*, x^*, y^*) are: $(0,0,1)$, $(\frac{1}{4}, \frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{1}{4}, -\frac{\sqrt{3}}{2}, -\frac{1}{2})$
- The Hessian is:

$$H(\lambda, x, y) = \begin{bmatrix} 0 & -2x & -2y \\ -2x & 2(y-1)^2 - 2\lambda & 4x(y-1) \\ -2y & 4x(y-1) & 2x^2 - 2\lambda \end{bmatrix}$$

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Constrained optimisation - equality

Example 2 - continuation

$$H(0,0,1) = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

- $det(H(0,0,1)) = 0$ - test is inconclusive

$$H(\frac{1}{4}, \frac{\sqrt{3}}{2}, \frac{1}{2}) = \begin{bmatrix} 0 & -\sqrt{3} & 1 \\ -\sqrt{3} & 4 & -3\sqrt{3} \\ 1 & -3\sqrt{3} & 1 \end{bmatrix} \quad H(\frac{1}{4}, -\frac{\sqrt{3}}{2}, \frac{1}{2}) = \begin{bmatrix} 0 & \sqrt{3} & 1 \\ \sqrt{3} & 4 & 3\sqrt{3} \\ 1 & 3\sqrt{3} & 1 \end{bmatrix}$$

- $det(H(\frac{1}{4}, \frac{\sqrt{3}}{2}, \frac{1}{2})) = det(H(\frac{1}{4}, -\frac{\sqrt{3}}{2}, \frac{1}{2})) = 11 > 0 \implies (\frac{1}{4}, \frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(\frac{1}{4}, -\frac{\sqrt{3}}{2}, -\frac{1}{2})$ are local maxima

Constrained optimisation - equality

Multiple constraints

- The presented method can be easily extended to the case of m-variable and n-constraints
- Consider the maximisation problem with $\boldsymbol{x} = (x1, .., x_n)$:

$$\max_{\boldsymbol{x}} f(\boldsymbol{x})$$
$$\text{s.t. } g_j(\boldsymbol{x}) = c_j \text{ for } j = 1, ..., m$$

- The Lagrangian for this problem is:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) - \sum_{j=1}^{n} \lambda_j (g_j(\boldsymbol{x}) - c_j)$$

- There is one Lagrange multiplier for each constraint
- The procedure is equivalent to the procedure for one constraint optimisation

Inequality constraints

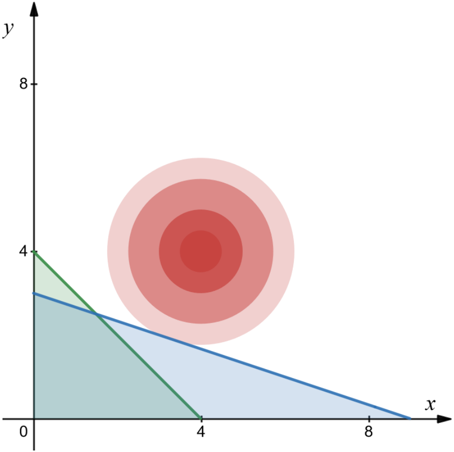
Introduction

- So far, we analysed only cases where constraints were in the form of equalities
- In economics a lot of optimisation problems are formulated with inequality constraints
- e.g. budget constraint: $\sum_{i=1}^{N} x_i p_i \leq m$
- We do not need to obligate a consumer to spend all of her budget

Inequality constraints

Introduction

- Consider the following example:

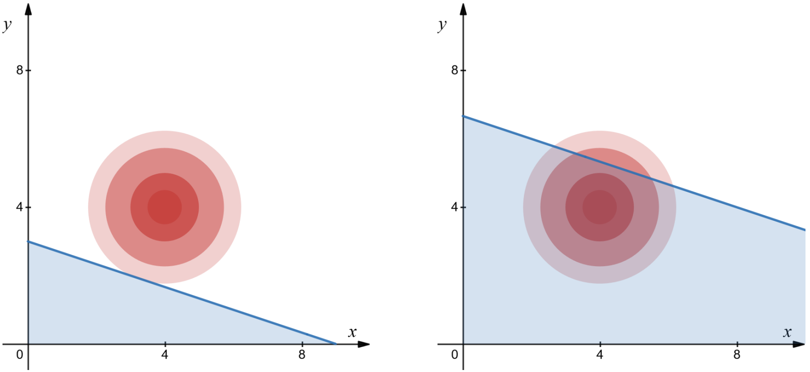


$$
\begin{aligned}
&\max_{x,y} f(x,y) \\
\text{s.t.}\quad & a_{1x}x + a_{1y} \leq c_1 \\
& a_{2x}x + a_{2y} \leq c_2 \\
& x, y \geq 0
\end{aligned}
$$

Inequality constraints

Binding vs slack

- At optimum, the constraint is either *binding* ($g(x^*) = c$) or *slack* ($g(x^*) < c$)

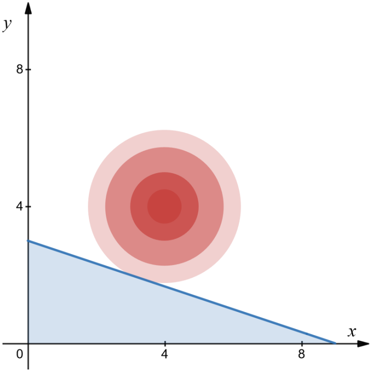


Binding constraint   Slack constraint

Inequality constraints

Binding

- In the case when the constraint is binding ($g(x^*) = c$):



- The constraint can be written as:

$$g(x^*) - c = 0$$

- All FOCs are satisfied:

$$L_i(\boldsymbol{x}, \lambda) = 0 \quad \forall i$$

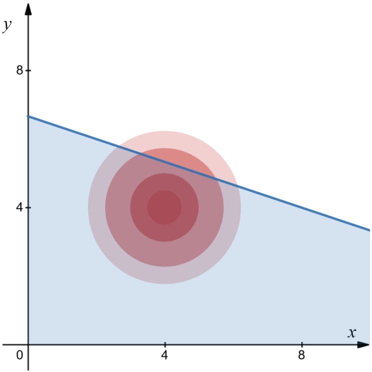
- λ has to be nonnegative:

$$\lambda \geq 0$$

Inequality constraints

Slack

- In the case when the constraint is slack ($g(x^*) < c$):



- The constraint problem reduces to unconstrained one
- All FOCs of $f(x)$ are satisfied:

$$f_i(\boldsymbol{x}, \lambda) = 0 \quad \forall i$$

- λ is zero:

$$\lambda = 0$$

Inequality constraints

KKT conditions

- Combining both cases, we can get set of sufficient conditions for maximisation problem
- Formally, the conditions (called Karush-Kuhn-Tucker (KKT) conditions) of the problem are:

$$\max_{\boldsymbol{x}} f(\boldsymbol{x})$$

$$\text{s.t. } g_j(\boldsymbol{x}) - c_j \leq 0 \text{ for } j = 1, , , .m$$

  - $L_i(\boldsymbol{x}, \lambda) = 0 \quad \forall i$
  - $\lambda_j \geq 0 \quad \forall j$
  - $g_j(\boldsymbol{x}) - c_j \leq 0 \quad \forall j$
  - $\lambda_j(g_j(\boldsymbol{x}) - c_j) = 0 \quad \forall j$
- In case of minimisation we have the same set of conditions, but use $\max_{\boldsymbol{x}} -f(\boldsymbol{x})$ instead of $\min_{\boldsymbol{x}} f(\boldsymbol{x})$

Inequality constraints

KKT conditions

- To solve a maximisation problem with inequality constraints:

1. Write down the Lagrangian:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) - \sum_{j=1}^{n} \lambda_j (g_j(\boldsymbol{x}) - c_j)$$

2. Equal all the first-order partial derivatives of $L(\boldsymbol{x}, \boldsymbol{\lambda})$ to 0
3. Implement the complementary and slackness conditions:

$$\lambda_j (g_j(\boldsymbol{x}) - c_j) = 0 \quad \forall i$$

4. Check if all λ's are nonnegative: $\lambda_j \geq 0 \quad \forall j$
5. Check if $\boldsymbol{x}$ satisfies constraints: $g_j(\boldsymbol{x}) - c_j \leq 0 \quad \forall j$

Inequality constraints

Example - maximisation problem

- Consider the following maximisation problem:

$$\max_{\boldsymbol{x}} -(x_1-4)^2-(x_2-4)^2$$
$$\text{s.t. } x_1+x_2\leq 4$$
$$x_1+3x_2\leq 9$$

- Then, formulate the Lagrangian:

$$L(x_1,x_2,\lambda_1,\lambda_2)=$$
$$=-(x_1-4)^2-(x_2-4)^2-\lambda_1(x_1+x_2-4)-\lambda_2(x_1+3x_2-9)$$

Inequality constraints

Example - FOCs

- Write down FOCs for the problem:

$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial x_1} = -2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial x_2} = -2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$
$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial \lambda_1} = x_1 + x_2 - 4 = 0$$
$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial \lambda_2} = x_1 + 3x_2 - 9 = 0$$

Inequality constraints

Example - complementary slackness conditions

- Write down complementary slackness conditions:

$$\lambda_1 (x_1 + x_2 - 4) = 0$$
$$\lambda_2 (x_1 + 3x_2 - 9) = 0$$

- It leads to four cases, which we need to analyse separately:
  - $\lambda_1 = \lambda_2 = 0$ which implies $x_1 + x_2 < 4$ and $x_1 + 3x_2 < 9$
  - $\lambda_1 > 0$ and $\lambda_2 = 0$ which implies $x_1 + x_2 = 4$ and $x_1 + 3x_2 < 9$
  - $\lambda_1 = 0$ and $\lambda_2 > 0$ which implies $x_1 + x_2 < 4$ and $x_1 + 3x_2 = 9$
  - $\lambda_1 > 0$ and $\lambda_2 > 0$ which implies $x_1 + x_2 = 4$ and $x_1 + 3x_2 = 9$

Inequality constraints

Example - case 1

- Case 1: $\lambda_1 = \lambda_2 = 0, \quad x_1 + x_2 < 4, \quad x_1 + 3x_2 < 9$
- FOCs becomes:

$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial x_1} = -2(x_1 - 4) = 0 \implies x_1 = 4$$
$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial x_2} = -2(x_2 - 4) = 0 \implies x_2 = 4$$

- Let's check if point $(4, 4)$ satisfies constraints:

$$x_1 + x_2 = 4 + 4 = 8 < 4$$

- We obtained contradiction, so point $(4, 4)$ cannot be a solution

Inequality constraints

Example - case 2

- Case 2: $\lambda_1 > 0, \quad \lambda_2 = 0, \quad x_1 + x_2 = 4, \quad x_1 + 3x_2 < 9$
- FOCs becomes:

$$\left.\begin{aligned} \frac{\partial L(x_1,x_2,\lambda_1,\lambda_2)}{\partial x_1} &= -2(x_1-4)-\lambda_1 = 0 \\ \frac{\partial L(x_1,x_2,\lambda_1,\lambda_2)}{\partial x_2} &= -2(x_2-4)-\lambda_1 = 0 \end{aligned}\right\} \implies x_1 = x_2$$

- Using constraint $x_1 + x_2 = 4$, we obtained a point $(2, 2)$
- Let's check if point $(2, 2)$ satisfies $x_1 + 3x_2 < 9$:

$$x_1 + x_2 = 2 + 2 = 4 < 9$$

- So, point $(2, 2)$ is a candidate for a solution

Inequality constraints

Example - case 3

- Case 3: $\lambda_1 = 0, \quad \lambda_2 > 0, \quad x_1 + x_2 < 4, \quad x_1 + 3x_2 = 9$
- FOCs becomes:

$$\begin{cases} \frac{\partial L(x_1,x_2,\lambda_1,\lambda_2)}{\partial x_1} & = -2(x_1 - 4) - \lambda_2 & = 0 \\ \frac{\partial L(x_1,x_2,\lambda_1,\lambda_2)}{\partial x_2} & = -2(x_2 - 4) - 3\lambda_2 & = 0 \\ \frac{\partial L(x_1,x_2,\lambda_1,\lambda_2)}{\partial \lambda_2} & = x_1 + 3x_2 - 9 & = 0 \end{cases}$$

- First two equations give us: $x_1 = \frac{1}{3}x_2 - \frac{8}{3}$
- Plugging into the last equation: $\frac{1}{3}x_2 - \frac{8}{3} + 3x_2 = 9 \implies x_2 = \frac{19}{10}\ x_1 = \frac{33}{10}$
- Plugging it into the constraint: $\frac{19}{10} + \frac{33}{10} = \frac{52}{10} \leq 4$ - contradiction

Inequality constraints

Example - case 4

- Case 4: $\lambda_1 > 0, \quad \lambda_2 > 0, \quad x_1 + x_2 = 4, \quad x_1 + 3x_2 = 9$
- FOCs becomes:

$$\begin{cases} \frac{\partial L(x_1,x_2,\lambda_1,\lambda_2)}{\partial x_1} & = -2(x_1 - 4) - \lambda_1 - \lambda_2 & = 0 \\ \frac{\partial L(x_1,x_2,\lambda_1,\lambda_2)}{\partial x_2} & = -2(x_2 - 4) - \lambda_1 - 3\lambda_2 & = 0 \\ \frac{\partial L(x_1,x_2,\lambda_1,\lambda_2)}{\partial \lambda_1} & = x_1 + x_2 - 4 & = 0 \\ \frac{\partial L(x_1,x_2,\lambda_1,\lambda_2)}{\partial \lambda_2} & = x_1 + 3x_2 - 9 & = 0 \end{cases}$$

- Combining last two equations give: $(x_1, x_2) = (\frac{3}{2}, \frac{5}{2})$
- Then the first two equations give: $\lambda_2 = -1 \geq 0$ - contradiction

Inequality constraints

Example - solution

- Taking all cases together, the only solution to the Kuhn-Tucker conditions is $(2, 2, 4, 0)$
- Hence, the unique solution of the problem is $(2, 2)$